

## Vector Analysis-Outline

### 1. Definition:

Scalar; symbole,  $m$ , example: mass, temperature, potential

vector; symbole,  $\bar{a}$ , example: force, velocity, displacement

### 2. Notation-Unit vector-rectangular coordinates

Right-hand system

$\bar{i}$ ,  $\bar{j}$ ,  $\bar{k}$  are unit vectors (also written  $i$ ,  $j$ ,  $k$ )

$$\bar{a} = ia_x + ja_y + ka_z$$

$$\bar{b} = ib_x + jb_y + kb_z$$

$$\bar{V} = iu_x + jv_y + kw_z$$

### 3. The dot or Scalar Product

$$\overline{AB} = AB \cos \theta$$

$$\overline{AB} = \overline{BA} \text{ (commutative law)}$$

$$\overline{A(\bar{B} + \bar{C})} = \overline{A\bar{B}} + \overline{A\bar{C}} \text{ (distributive law)}$$

$$m(\overline{AB}) = (m\bar{A})\bar{B} = \bar{A}(m\bar{B}) = (\overline{AB})m$$

$$ii = jj = kk = 1, \quad ij = jk = ki = 0$$

$$\text{if } \bar{A} = iA_x + jA_y + kA_z, \quad \bar{B} = iB_x + jB_y + kB_z$$

$$C = \overline{AB} = A_xB_x + A_yB_y + A_zB_z, \quad |C| = AB \cos \theta$$

$$\overline{AA} = A^2 = A_x^2 + A_y^2 + A_z^2$$

$$\overline{BB} = B^2 = B_x^2 + B_y^2 + B_z^2$$

### 4. Cross Product

$$\bar{D} = \bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \quad |D| = AB \sin \theta$$

$$\text{where } \bar{A} = iA_x + jA_y + kA_z, \quad \bar{B} = iB_x + jB_y + kB_z$$

$$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A} \text{ (commutative law fails)}$$

$$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C} \text{ (distributive law)}$$

$$m(\bar{A} \times \bar{B}) = m(\bar{A}) \times \bar{B} = \bar{A}m(\times \bar{B}) = (\bar{A} \times \bar{B})m$$

$$i \times i = j \times j = k \times k = 0, \quad i \times j = k, \quad j \times k = i, \quad k \times i = j$$

## 5. Differentiation Fofrmulas

$$\frac{d}{du}(\bar{A} + \bar{B}) = \frac{d\bar{A}}{du} + \frac{d\bar{B}}{du}$$

$$\frac{d}{du}(\bar{A}\bar{B}) = \bar{A} \frac{d\bar{B}}{du} + \frac{d\bar{A}}{du} \bar{B}$$

$$\frac{d}{du}(\bar{A} \times \bar{B}) = \bar{A} \times \frac{d\bar{B}}{du} + \frac{d\bar{A}}{du} \times \bar{B}$$

$$\frac{d}{du}(\phi \times \bar{A}) = \phi \frac{d\bar{B}}{du} + d \frac{\phi}{du} \bar{A}$$

$$\frac{d}{du}(\bar{A}\bar{B} \times \bar{C}) = \bar{A}\bar{B} \times \frac{d\bar{C}}{du} + \bar{A} \frac{d\bar{B}}{du} \times \bar{C} + \frac{d\bar{A}}{du} \bar{B} \times \bar{C}$$

$$\frac{d}{du}\{\bar{A} \times (\bar{B} \times \bar{C})\} = \bar{A} \times (\bar{B} \times \frac{d\bar{C}}{du}) + \bar{A} \times (\frac{d\bar{B}}{du} \times \bar{C}) + \frac{d\bar{A}}{du} \times (\bar{B} \times \bar{C})$$

The order in these products may be important.

## 6. Gradient, Divergence and Curl (Rotation)

The vector differential operator-Del

$$\text{Define : } \nabla \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

(The Gradient)

$$\nabla \phi \text{ (grad}\phi) = (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k) \phi = (\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k) \phi$$

Note that  $\nabla \phi$  defines a vector field.

(The Divergence)

$$\text{Let } \bar{V} = V_x i + V_y j + V_z k$$

$$\nabla \bar{V} = (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k)(V_x i + V_y j + V_z k)$$

$$\frac{\partial V_x}{\partial x} i + \frac{\partial V_y}{\partial y} j + \frac{\partial V_z}{\partial z} k$$

Note that  $\nabla \bar{V} \neq \bar{V} \nabla$

(The Curl or Rotation)

$$\text{Let } \bar{V} = V_x i + V_y j + V_z k$$

$$\nabla \times \bar{V} = \left( \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \times (V_x i + V_y j + V_z k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Example 1. (Gradient)

If  $\phi(x, y, z) = 3x^2y - y^3z^2$ , find  $\nabla\phi$  at the point  $(1, -2, -1)$ .

(Sol.)

$$\nabla\phi = \left( \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) (3x^2y - y^3z^2)$$

$$= 6xyi + (3x^2 - 3y^2z^2)j - 2y^3zk$$

$$\text{At point } (1, -2, -1), \nabla\phi = -12i - 9j - 16k$$

Example 2. (Divergence)

(Sol.) Find  $\nabla\phi$  if  $\phi = \ln|\bar{r}|$ .

$$\text{where } \bar{r} = xi + yj + zk$$

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \ln|\bar{r}| = \frac{1}{2} \ln(x^2 + y^2 + z^2)$$

$$\nabla\phi = \frac{1}{2} \left\{ i \frac{2x}{x^2 + y^2 + z^2} + j \frac{2y}{x^2 + y^2 + z^2} + k \frac{2z}{x^2 + y^2 + z^2} \right\}$$

$$= \frac{xi + yj + zk}{x^2 + y^2 + z^2} = \frac{|\bar{r}|}{r^2}$$

Example 3. (Divergence)

Given  $\phi = 2x^3y^2z^4$ , find  $\nabla\phi$  ( $\text{div grad}\phi$ ).

(Sol.)

$$\nabla\phi = i \frac{\partial}{\partial x} (2x^3y^2z^4) + j \frac{\partial}{\partial y} (2x^3y^2z^4) + \frac{\partial}{\partial z} (2x^3y^2z^4)$$

$$= 6x^2y^2z^4i + 4x^3yz^4j + 8x^3y^2z^3k$$

$$\nabla\phi = \frac{\partial}{\partial x} (6x^2y^2z^4) + \frac{\partial}{\partial y} (4x^3yz^4) + \frac{\partial}{\partial z} (8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Example 4. (Curl or Rot)

If  $\bar{A} = x^2yi - 2xzj + 2yzk$ , find  $\text{curl curl } \bar{A}$ .

(Sol.)

$$\begin{aligned}\nabla \times (\nabla \times \bar{A}) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix} \\ &= \nabla \times \{(2x + 2z)i - (x^2 + 2z)k\} \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix} = 2(x + 1)j\end{aligned}$$